

EXCEPTIONAL  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -SYMMETRIC SPACES

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ABSTRACT. The notion of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces is a generalization of classical symmetric spaces, where the group  $\mathbb{Z}_2$  is replaced by  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . In this article, a classification is given of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces  $G/K$  where  $G$  is an exceptional compact Lie group or  $\text{Spin}(8)$ , complementing recent results of Bahturin and Goze. Our results are equivalent to a classification of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -gradings on the exceptional simple Lie algebras  $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$  and  $\mathfrak{so}(8)$ .

## 1. INTRODUCTION AND RESULTS

The notion of  $\Gamma$ -symmetric spaces introduced by Lutz [13] is a generalization of the classical notion of a symmetric space.

**Definition 1.1.** Let  $\Gamma$  be a finite abelian group and let  $G$  be a connected Lie group. A homogeneous space  $G/K$  is called  $\Gamma$ -symmetric if  $G$  acts almost effectively on  $G/K$  and there is an injective homomorphism  $\rho: \Gamma \rightarrow \text{Aut } G$ , such that  $G_0^\Gamma \subseteq K \subseteq G^\Gamma$ , where  $G^\Gamma$  is the subgroup of elements fixed by  $\rho(\Gamma)$  and  $G_0^\Gamma$  its connected component.

In the case  $\Gamma = \mathbb{Z}_2$  this is just the classical definition of symmetric spaces, in case  $\Gamma = \mathbb{Z}_k$  one obtains  $k$ -symmetric spaces, as studied in [6]. In case  $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$  we can rephrase the definition as follows. A homogeneous space  $G/K$  is  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric if and only if there are two different commuting involutions on  $G$ , i.e. there are  $\sigma, \tau \in \text{Aut } G \setminus \{\text{id}_G\}$  such that  $\sigma^2 = \tau^2 = \text{id}_G$ ,  $\sigma \neq \tau$  and  $\sigma\tau = \tau\sigma$  such that  $(G^\sigma \cap G^\tau)_0 \subseteq K \subseteq G^\sigma \cap G^\tau$ .

It is the purpose of this paper to give a classification of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces in case  $G$  is a simply connected compact Lie group of isomorphism type  $\text{Spin}(8)$ ,  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$  or  $G_2$ . This amounts to a classification of pairs of commuting involutions on the Lie algebras of these groups. Recently, such a classification has been obtained by Bahturin and Goze [1] for the case of simple classical Lie algebras except  $\mathfrak{so}(8)$ . The results of the classification in the exceptional and  $\text{Spin}(8)$  case are given in Theorems 1.2 and 1.3, respectively, below.

Let  $G/K$  be a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space. Then there is a triple of involutions  $\sigma, \tau, \sigma\tau$  of  $G$  and we say that the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of *type*  $(X, Y, Z)$  where  $X, Y, Z$  denote the local isomorphism classes of the symmetric spaces  $G/G^\sigma, G/G^\tau$  and  $G/G^{\sigma\tau}$ , respectively. For example, we will show that there are two commuting involutions  $\sigma, \tau$  on  $G = E_7$  such that  $G/G^\sigma, G/G^\tau$  and  $G/G^{\sigma\tau}$  are isomorphic to symmetric spaces of type E V, E VI and E VII, respectively and such that the Lie algebra of  $E_6^\sigma \cap E_6^\tau$  is isomorphic to  $\mathfrak{su}(6) + \mathfrak{sp}(1) + \mathbb{R}$ , cf. Table 2. Thus we say that the corresponding  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type (E V, E VI, E VII), abbreviated as E V-VI-VII.

The problem of commuting involutions has a geometric interpretation in terms of classical  $\mathbb{Z}_2$ -symmetric spaces. Let  $G$  be a simple compact connected Lie group and let  $H, L$  be symmetric subgroups of  $G$  (see Definition 2.1 below) such that  $(G^\sigma)_0 \subseteq H \subseteq G^\sigma$  and  $(G^\tau)_0 \subseteq L \subseteq G^\tau$ , where  $\sigma, \tau \in \text{Aut}(G)$  are involutions. Then  $M = G/L$  endowed with a left  $G$ -invariant metric is a symmetric space in the classical sense and there is a natural isometric action of  $H$  on  $M$

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2000 *Mathematics Subject Classification.* 53C30, 53C35; 17B40.

*Key words and phrases.* exceptional  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space, Lie algebra grading.

Type	$\mathfrak{g}$	$\mathfrak{k}$
E I-I-II	$\mathfrak{e}_6$	$\mathfrak{so}(6) + \mathbb{R}$
E I-I-III	$\mathfrak{e}_6$	$\mathfrak{sp}(2) + \mathfrak{sp}(2)$
E I-II-IV	$\mathfrak{e}_6$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$
E II-II-II	$\mathfrak{e}_6$	$\mathfrak{su}(3) + \mathfrak{su}(3) + \mathbb{R} + \mathbb{R}$
E II-II-III	$\mathfrak{e}_6$	$\mathfrak{su}(4) + \mathfrak{sp}(1) + \mathfrak{sp}(1) + \mathbb{R}$
E II-III-III	$\mathfrak{e}_6$	$\mathfrak{su}(5) + \mathbb{R} + \mathbb{R}$
E III-III-III	$\mathfrak{e}_6$	$\mathfrak{so}(8) + \mathbb{R} + \mathbb{R}$
E III-IV-IV	$\mathfrak{e}_6$	$\mathfrak{so}(9)$
E V-V-V	$\mathfrak{e}_7$	$\mathfrak{so}(8)$
E V-V-VI	$\mathfrak{e}_7$	$\mathfrak{su}(4) + \mathfrak{su}(4) + \mathbb{R}$
E V-V-VII	$\mathfrak{e}_7$	$\mathfrak{sp}(4)$
E V-VI-VII	$\mathfrak{e}_7$	$\mathfrak{su}(6) + \mathfrak{sp}(1) + \mathbb{R}$
E VI-VI-VI	$\mathfrak{e}_7$	$\mathfrak{so}(8) + \mathfrak{so}(4) + \mathfrak{sp}(1)$
	$\mathfrak{e}_7$	$\mathfrak{u}(6) + \mathbb{R}$
E VI-VII-VII	$\mathfrak{e}_7$	$\mathfrak{so}(10) + \mathbb{R} + \mathbb{R}$
E VII-VII-VII	$\mathfrak{e}_7$	$\mathfrak{f}_4$
E VIII-VIII-VIII	$\mathfrak{e}_8$	$\mathfrak{so}(8) + \mathfrak{so}(8)$
E VIII-VIII-IX	$\mathfrak{e}_8$	$\mathfrak{su}(8) + \mathbb{R}$
E VIII-IX-IX	$\mathfrak{e}_8$	$\mathfrak{so}(12) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$
E IX-IX-IX	$\mathfrak{e}_8$	$\mathfrak{e}_6 + \mathbb{R} + \mathbb{R}$
F I-I-I	$\mathfrak{f}_4$	$\mathfrak{u}(3) + \mathbb{R}$
F I-I-II	$\mathfrak{f}_4$	$\mathfrak{sp}(2) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$
F II-II-II	$\mathfrak{f}_4$	$\mathfrak{so}(8)$
G	$\mathfrak{g}_2$	$\mathbb{R} + \mathbb{R}$

TABLE 1. Exceptional  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces.

which is hyperpolar, i.e. there exists a flat submanifold which meets all the orbits and such that its intersections with the orbits are everywhere orthogonal, see [10] and [12]. These actions have first been considered by Hermann [8]; they are called *Hermann actions*.

In [9], Hermann proved that the  $H$ -orbit through  $gL \in M$ ,  $g \in G$  is totally geodesic if and only if the involutions  $i_g \sigma i_g^{-1}$  and  $\tau$  commute where we use the notation  $i_g$  to denote the automorphism of a Lie group  $G$  given by  $i_g(x) = gxg^{-1}$ ; this means that the classification of commuting involutions on  $G$  is equivalent to a classification of totally geodesic orbits of Hermann actions.

Conlon [3] determined all pairs of conjugacy classes of involutions  $\sigma, \tau$  for which there is a  $g \in G$  such that  $i_g \sigma i_g^{-1}$  and  $\tau$  commute. The proof relies on the unpublished notes [2]. (For inner involutions  $\sigma = i_a$ ,  $\tau = i_b$  there is obviously always such a  $g$  since there is a  $g \in G$  such that  $ga g^{-1}$  and  $b$  are contained in one and the same maximal torus of  $G$ .) Our classification however uses a direct approach which does not rely on any of the previously mentioned results.

**Theorem 1.2.** *Let  $\mathfrak{g}$  be a compact exceptional Lie algebra of type  $\mathfrak{e}_6$ ,  $\mathfrak{e}_7$ ,  $\mathfrak{e}_8$ ,  $\mathfrak{f}_4$ , or  $\mathfrak{g}_2$ . If  $\sigma, \tau \in \text{Aut}(\mathfrak{g})$  are two commuting involutions such that  $\sigma \neq \tau$  then the pair  $(\mathfrak{g}, \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau)$  is one of the pairs  $(\mathfrak{g}, \mathfrak{k})$  given by Table 1; the conjugacy classes of  $\sigma$ ,  $\tau$  and  $\sigma\tau$  are given by the first row of Table 1. Conversely, for any pair of Lie algebras  $(\mathfrak{g}, \mathfrak{k})$  in Table 1 there exists a pair  $(\sigma, \tau)$  of commuting involutions of  $\mathfrak{g}$  such that  $\mathfrak{k} \cong \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$ .*

The data in Table 1 in all cases but one determines the conjugacy class of the subalgebra  $\mathfrak{k} \subset \mathfrak{g}$ , see Remark 4.2. To prove Theorem 1.2, we first exhibit various standard examples for pairs of involutions on exceptional Lie algebras in Section 3. Then it is shown in Section 4 that these constructions already exhaust all possibilities for  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces.

**Theorem 1.3.** *Let  $\mathfrak{k} \subset \mathfrak{g} = \mathfrak{so}(8)$  be a subalgebra such that there are two commuting involutions  $\sigma$  and  $\tau$ ,  $\sigma \neq \tau$ , of  $\mathfrak{g}$  with  $\mathfrak{k} = \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$ . Then there is an automorphism  $\varphi$  of  $\mathfrak{so}(8)$  such that  $\varphi(\mathfrak{k})$  is one of the following subalgebras of  $\mathfrak{so}(8)$ .*

- (i)  $\mathfrak{so}(n_1) + \mathfrak{so}(n_2) + \mathfrak{so}(n_3)$ ,  $n_1 + n_2 + n_3 = 8$ ,  $n_i \geq 1$ ;
- (ii)  $\mathfrak{so}(n_1) + \mathfrak{so}(n_2) + \mathfrak{so}(n_3) + \mathfrak{so}(n_4)$ ,  $n_1 + n_2 + n_3 + n_4 = 8$ ,  $n_i \geq 1$ ;
- (iii)  $\mathfrak{u}(3) + \mathfrak{u}(1)$ .

Note that for the case of  $\mathfrak{so}(8)$  there are less cases than there are in the classification of Bahturin and Goze [1] for the Lie algebras  $\mathfrak{so}(n)$ ,  $n \neq 8$ , since some of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric subalgebras for general  $\mathfrak{so}(n)$  which appear as distinct cases in [1] are conjugate by some automorphism of  $\mathfrak{so}(8)$ . Theorem 1.3 is proved in Section 5.

## 2. PRELIMINARIES

**Definition 2.1.** Let  $\mathfrak{g}$  be a Lie algebra. We say that  $\mathfrak{k} \subset \mathfrak{g}$  is a *symmetric subalgebra* of  $\mathfrak{g}$  if there is a nontrivial automorphism  $\sigma$  of  $\mathfrak{g}$  with  $\sigma^2 = \text{id}_{\mathfrak{g}}$  such that  $\mathfrak{k} = \mathfrak{g}^\sigma := \{X \in \mathfrak{g} \mid \sigma(X) = X\}$ . If  $G$  is a Lie group, we say that a closed subgroup  $K \subset G$  is a *symmetric subgroup* if the Lie algebra of  $K$  is a symmetric subalgebra in the Lie algebra of  $G$ .

**Proposition 2.2.** Let  $\mathfrak{g}$  be a real Lie algebra. Let  $\sigma, \tau$  be automorphisms of  $\mathfrak{g}$  such that  $\sigma^2 = \tau^2 = \text{id}_{\mathfrak{g}}$ ,  $\sigma\tau = \tau\sigma$ ,  $\sigma \neq \tau$ . Then the Lie algebra  $\mathfrak{g}$  splits as a direct sum of vector spaces

$$(2.1) \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_\sigma \oplus \mathfrak{g}_\tau \oplus \mathfrak{g}_{\sigma\tau},$$

where  $\mathfrak{g}_1 = \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$ ,  $\mathfrak{g}_1 + \mathfrak{g}_\sigma = \mathfrak{g}^\sigma$ ,  $\mathfrak{g}_1 + \mathfrak{g}_\tau = \mathfrak{g}^\tau$ ,  $\mathfrak{g}_1 + \mathfrak{g}_{\sigma\tau} = \mathfrak{g}^{\sigma\tau}$  such that the following hold:

- (i) We have  $[\mathfrak{g}_\varphi, \mathfrak{g}_\psi] \subseteq \mathfrak{g}_{\varphi\psi}$  for all  $\varphi, \psi \in \Gamma$ , i.e. the Lie algebra  $\mathfrak{g}$  is  $\Gamma$ -graded, where  $\Gamma = \{1, \sigma, \tau, \sigma\tau\}$  is the subgroup of  $\text{Aut}(\mathfrak{g})$  generated by  $\sigma$  and  $\tau$ .
- (ii) For all  $\varphi \in \Gamma \setminus \{1\}$  we have that  $\mathfrak{g}_1$  is a symmetric subalgebra of  $\mathfrak{g}_1 + \mathfrak{g}_\varphi$ .
- (iii) For all  $\varphi \in \Gamma \setminus \{1\}$  we have that  $\mathfrak{g}_1 + \mathfrak{g}_\varphi$  is a symmetric subalgebra of  $\mathfrak{g}$ .

*Proof.* Consider the Cartan decomposition  $\mathfrak{g} = \mathfrak{g}^\sigma + \mathfrak{p}^\sigma$ . Since  $\tau$  commutes with  $\sigma$ , it leaves this decomposition invariant and hence we have further splittings  $\mathfrak{g}^\sigma = (\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau) + (\mathfrak{g}^\sigma \cap \mathfrak{p}^\tau)$  and  $\mathfrak{p}^\sigma = (\mathfrak{p}^\sigma \cap \mathfrak{g}^\tau) + (\mathfrak{p}^\sigma \cap \mathfrak{p}^\tau)$ , where  $\mathfrak{g} = \mathfrak{g}^\tau + \mathfrak{p}^\tau$  is the Cartan decomposition with respect to  $\tau$ . Define  $\mathfrak{g}_\sigma = \mathfrak{g}^\sigma \cap \mathfrak{p}^\tau$ ,  $\mathfrak{g}_\tau = \mathfrak{g}^\tau \cap \mathfrak{p}^\sigma$  and  $\mathfrak{g}_{\sigma\tau} = \mathfrak{p}^\sigma \cap \mathfrak{p}^\tau$ . Now the assertions of the proposition are easily checked.  $\square$

Conversely, given a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading on a Lie algebra  $\mathfrak{g}$ , one can define a pair of commuting involutions  $\sigma, \tau$  on  $\mathfrak{g}$  such that (2.1) agrees with the grading. Thus our results are equivalent to a classification of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -gradings on  $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$  and  $\mathfrak{so}(8)$ .

In order to classify  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces, we want to find all possibilities for a pair of involutions  $(\sigma, \tau)$  such that  $\sigma$  and  $\tau$  commute. It follows from Proposition 2.2 that in this case we have

$$(2.2) \quad \dim \mathfrak{g} - \dim \mathfrak{g}^\sigma - \dim \mathfrak{g}^\tau = \dim \mathfrak{g}^{\sigma\tau} - 2 \dim \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau.$$

This gives a necessary condition for the existence of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces of certain types. Note that the left hand side in (2.2) depends only on the conjugacy classes of  $\sigma$  and  $\tau$ ; we define  $d := d(\sigma, \tau) := \dim \mathfrak{g} - \dim \mathfrak{g}^\sigma - \dim \mathfrak{g}^\tau$ . In order to find candidates for the pair  $(\mathfrak{g}^{\sigma\tau}, \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau)$ , we list all symmetric subalgebras of symmetric subalgebras in exceptional compact Lie algebras in Table 4,

where for each pair  $(\mathfrak{h}, \mathfrak{k})$  the number  $c(\mathfrak{h}, \mathfrak{k}) := \dim \mathfrak{h} - 2 \dim \mathfrak{k}$  is given. By (2.2), the only candidates for a pair  $(\mathfrak{g}^{\sigma\tau}, \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau)$  are those pairs  $(\mathfrak{h}, \mathfrak{k})$ , where the number  $c(\mathfrak{h}, \mathfrak{k})$  equals  $d$ . Furthermore, we may eliminate immediately all pairs  $(\mathfrak{h}, \mathfrak{k})$  from the list of candidates which do not fulfil the necessary condition that  $\mathfrak{g}^\sigma$  and  $\mathfrak{g}^\tau$  both contain a symmetric subalgebra isomorphic to  $\mathfrak{k}$ . However, these are only necessary conditions and we have to check in each case if a decomposition (2.1) with  $\mathfrak{g}^{\sigma\tau} \cong \mathfrak{h}$ ,  $\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau \cong \mathfrak{k}$  actually exists.

For the convenience of the reader we list the symmetric subalgebras  $\mathfrak{k}$  of simple compact exceptional Lie algebras  $\mathfrak{g}$  below in Table 2; in Tables 3 and 5 we list pairs of (conjugacy classes of) involutions on the exceptional Lie algebras and  $\mathfrak{so}(8)$ , respectively, together with the numbers  $d = d(\sigma, \tau)$ .

	$\mathfrak{g}$	$\mathfrak{k}$		$\mathfrak{g}$	$\mathfrak{k}$
E I	$\mathfrak{e}_6$	$\mathfrak{sp}(4)$	E VII	$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{R}$
E II	$\mathfrak{e}_6$	$\mathfrak{su}(6) + \mathfrak{sp}(1)$	E VIII	$\mathfrak{e}_8$	$\mathfrak{so}(16)$
E III	$\mathfrak{e}_6$	$\mathfrak{so}(10) + \mathbb{R}$	E IX	$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$
E IV	$\mathfrak{e}_6$	$\mathfrak{f}_4$	F I	$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$
E V	$\mathfrak{e}_7$	$\mathfrak{su}(8)$	F II	$\mathfrak{f}_4$	$\mathfrak{so}(9)$
E VI	$\mathfrak{e}_7$	$\mathfrak{so}(12) + \mathfrak{sp}(1)$	G	$\mathfrak{g}_2$	$\mathfrak{sp}(1) + \mathfrak{sp}(1)$

TABLE 2. Exceptional symmetric spaces of type I.

### 3. INVOLUTIONS OF REDUCTIVE LIE ALGEBRAS AND CONSTRUCTION OF EXAMPLES

We will now discuss some constructions of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces. We start with a list of several possibilities how one can define involutive automorphisms of a reductive complex Lie algebra from a given root space decomposition. Let  $\mathfrak{g}$  be a reductive complex Lie algebra with Cartan subalgebra  $\mathfrak{g}_0$  and let

$$(3.1) \quad \mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

be the root space decomposition with respect to  $\mathfrak{g}_0$ . Also assume that we have chosen once and for all a set  $\{\alpha_1, \dots, \alpha_n\}$  of simple roots.

**Type 1.** Let  $\mathfrak{h} \subset \mathfrak{g}$  be a symmetric subalgebra of maximal rank. Then we may assume that  $\mathfrak{h}$  contains  $\mathfrak{g}_0$ . Hence there is a subset  $S \subset \Phi$  such that  $\mathfrak{h} = \mathfrak{g}_0 + \sum_{\alpha \in S} \mathfrak{g}_\alpha$  and we may define an involutive automorphism  $\sigma$  of  $\mathfrak{g}$  by requiring that  $\sigma(X) = X$  if  $X \in \mathfrak{h}$  and  $\sigma(X) = -X$  if  $X \in \sum_{\alpha \in \Phi \setminus S} \mathfrak{g}_\alpha$ . The corresponding symmetric spaces are exactly those where the involution is an inner automorphism.

**Type 2.** Outer involutions induced from automorphisms of the Dynkin diagram, see [7], Ch. X, § 5: The corresponding simply connected irreducible symmetric spaces with simple compact isometry group are  $\mathrm{SU}(2n+1)/\mathrm{SO}(2n+1)$ ,  $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ ,  $\mathrm{SO}(2n+2)/\mathrm{SO}(2n+1)$  and  $\mathrm{E}_6/\mathrm{F}_4$ .

**Type 3.** Let  $\mathfrak{g}$  be a reductive complex Lie algebra and let  $\mathfrak{t} \subset \mathfrak{g}$  be a Cartan subalgebra. Then there is an automorphism  $\sigma$  of  $\mathfrak{g}$  which acts as minus identity on  $\mathfrak{t}$  and sends each root to its negative. The symmetric spaces  $G/H$  given in this way are exactly those where  $\mathrm{rk} G/H = \mathrm{rk} G$ ; the Satake diagram of  $G/H$  is given by the Dynkin diagram of  $G$  with uniform multiplicity one. The automorphisms obtained in this way may be inner or outer. The corresponding irreducible compact symmetric space are: A I,  $\mathrm{SO}(2n+1)/\mathrm{SO}(n+1) \times \mathrm{SO}(n)$ ,  $\mathrm{Sp}(n)/\mathrm{U}(n)$ ,  $\mathrm{SO}(2n)/\mathrm{SO}(n) \times \mathrm{SO}(n)$ , E I, E V, E VIII, F I, G.

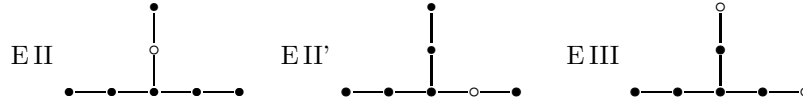
Note that the distinction of the three types of involutions above pertains to the action of the involution with respect to one fixed Cartan subalgebra. While Type 1 involutions obviously are never conjugate to involutions of type Type 2, a Type 3 involution may be conjugate to a Type 1 or Type 2 involution.

In the examples below, we show how to construct various  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces mainly by combining two commuting involutions as defined above with respect to one fixed root space decomposition (3.1). We start with the examples where both involutions are of Type 1.

**Example 3.1.** Let  $G$  be a connected compact Lie group and let  $\sigma_1$  and  $\sigma_2$  be two involutions of  $G$  defined by conjugation with the elements  $g_1$  and  $g_2$ , respectively, of  $G$ , such that the fixed point sets  $G^{\sigma_1}$  and  $G^{\sigma_2}$  are non-isomorphic. After conjugation, we may assume that  $g_1$  and  $g_2$  are both contained in one and the same maximal Torus  $T$  of  $G$ ; in particular,  $\sigma_1$  and  $\sigma_2$  commute. Consider the root space decomposition (3.1) where  $\mathfrak{g}_0$  is the complexification of the Lie algebra of  $T$ . Then the automorphisms of  $\mathfrak{g}_0$  induced by  $\sigma$  and  $\tau$  are both of Type 1 and the complexification of the Lie algebra of  $G^\sigma \cap G^\tau$  is given by  $\mathfrak{g}_0 + \sum_{\alpha \in S_1 \cap S_2} \mathfrak{g}_\alpha$ , where  $S_i$  is the root system of  $G^{\sigma_i}$ ,  $i = 1, 2$ , w.r.t.  $\mathfrak{g}_0$ .

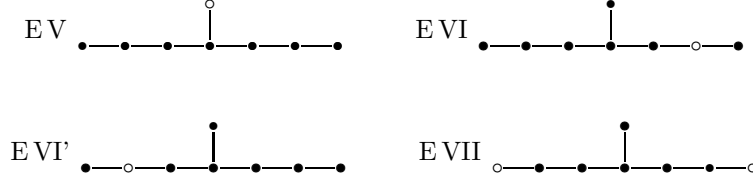
The maximal subgroups of maximal rank in a simple compact Lie group  $G$  can be conveniently described by using extended Dynkin diagrams, cf. [14], 1.3.11. The nodes of the extended Dynkin diagram correspond to roots  $\alpha_0, \alpha_1, \dots, \alpha_n$ , where  $\alpha_1, \dots, \alpha_n$  are the simple roots and  $\alpha_0 = -\delta$  where  $\delta$  is highest root. The simple roots of a maximal subgroup of maximal rank are then given by certain subsets of the set of nodes of the extended Dynkin diagram. For the groups  $E_6$ ,  $E_7$  and  $E_8$  these subsets are given below by the black nodes  $\bullet$  for the symmetric ones among the subgroups of maximal rank. The root systems  $S_i$  of these subgroups are given as the union of all roots which are integral linear combinations of the simple roots corresponding to black nodes. Now we may consider two diagrams depicting two symmetric subgroups  $G^{\sigma_1}$  and  $G^{\sigma_2}$  of maximal rank. Then the root system of  $G^{\sigma_1} \cap G^{\sigma_2}$  with respect to  $\mathfrak{g}_0$  is given by  $S_1 \cap S_2$  and it is a straightforward task to explicitly determine this set. However, to completely avoid these computations, we will use a simplified approach. We know from [14], 1.3, Proposition 15, that both  $G^{\sigma_1}$  and  $G^{\sigma_2}$  contain a subgroup whose simple roots with respect to  $\mathfrak{g}_0$  are those whose corresponding nodes are marked black in both diagrams. Thus we obtain the set of simple roots of a regular subgroup of  $G$  contained in the intersection  $G^\sigma \cap G^\tau$ . This information, together with a dimension count using Tables 3 and 4, turns out to be sufficient in all cases given below to identify the connected component of the intersection  $G^\sigma \cap G^\tau$ , which is a symmetric subgroup of both  $G^\sigma$  and  $G^\tau$ .

The symmetric subgroups of maximal rank in  $E_6$ , namely  $SU(6) \cdot Sp(1)$  and  $Spin(10) \cdot U(1)$  are given by the following diagrams:



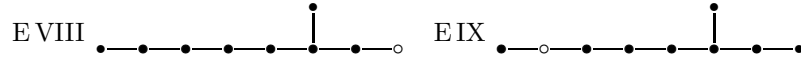
Comparing E-II and E-III, we see that there is a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space  $G/K$  such that  $\mathfrak{k}$  contains a subalgebra isomorphic to  $\mathfrak{su}(5) + \mathbb{R} + \mathbb{R}$ . A simple dimension count, facilitated by Tables 3 and 4, shows that the corresponding  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type E II-III-III with  $\mathfrak{k} \cong \mathfrak{su}(5) + \mathbb{R} + \mathbb{R}$ . Using an alternative embedding of  $SU(6) \cdot Sp(1)$  into  $E_6$ , shown as E II', and combining it with E II, we obtain a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space with  $\mathfrak{k} \cong \mathfrak{su}(4) + \mathfrak{sp}(1) + \mathfrak{sp}(1) + \mathbb{R} \cong \mathfrak{so}(6) + \mathfrak{so}(4) + \mathbb{R}$  of type E II-II-III. Combining E III and E III' (as given in Example 3.6 below) we construct a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E III-III-III with  $\mathfrak{k} = \mathfrak{so}(8) + \mathbb{R} + \mathbb{R}$ .

The connected symmetric subgroups of  $E_7$ , namely  $SU(8)$ ,  $Spin(12) \cdot Sp(1)$  and  $E_6 \cdot U(1)$ , are given by the following diagrams.



Combining E V and E VI we obtain a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space with  $\mathfrak{k} = \mathfrak{su}(6) + \mathfrak{sp}(1) + \mathbb{R}$  of type E V-VI-VII. Combining E VI and E VII we obtain a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space with  $\mathfrak{k}$  containing a subalgebra isomorphic to  $\mathfrak{so}(10) + \mathbb{R} + \mathbb{R}$ ; using Tables 3 and 4 it follows that the corresponding  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type E VI-VII-VII with  $\mathfrak{k} \cong \mathfrak{so}(10) + \mathbb{R} + \mathbb{R}$ . Combining E VI with E VI', we obtain a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space such that  $\mathfrak{k}$  contains a subalgebra isomorphic to  $\mathfrak{so}(8) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$ . We see from Tables 3 and 4 that the corresponding  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type E VI-VI-VI with  $\mathfrak{k} \cong \mathfrak{so}(8) + \mathfrak{so}(4) + \mathfrak{sp}(1)$ .

The connected symmetric subgroups of  $E_8$ , namely  $SO'(16)$  and  $E_7 \cdot Sp(1)$  are given by the following diagrams.

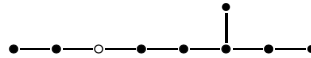


Combining the two diagrams and using Tables 3 and 4 proves the existence of a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E VIII-IX-IX with  $\mathfrak{k} \cong \mathfrak{so}(12) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$ .

**Example 3.2.** Let now  $G$  be simple and let  $g \in G$  be an element with  $i_g^2 = \text{id}_G$ . Let  $T$  be a maximal torus of  $G$  containing  $g$ . Let  $w = k Z_G(T)$ ,  $k \in N_G(T)$ , be an element of the Weyl group  $W_G = N_G(T)/Z_G(T)$  of  $G$  such that the action of  $w$  on  $T$  does not leave the root system of  $Z_G(\{g\})$  invariant. Define  $h = w \cdot g = k g k^{-1}$ . Then  $i_g$  and  $i_h$  are two conjugate commuting involutions of  $G$  whose fixed point sets do not agree. Hence they define a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space.

We will now use this construction to prove the existence of various  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces. First, let  $G = F_4$  and let  $i_g$  be an inner involution of type F II, i.e. the connected component of the fixed point set of  $i_g$  is isomorphic to  $\text{Spin}(9)$ . Define another involution  $i_h$  as above; it is also of type F II, but the fixed point sets of  $i_g$  and  $i_h$  do not agree. From Table 3 we read off that we have  $d = -20$  for a pair of involutions of Type F II-II. In Table 4 there are two entries with  $\mathfrak{g} = \mathfrak{f}_4$  and  $c(\mathfrak{h}, \mathfrak{k}) = -20$ , namely  $\mathfrak{k} = \mathfrak{sp}(3) + \mathbb{R}$  and  $\mathfrak{k} = \mathfrak{so}(8)$ . Since  $\mathfrak{sp}(3)$  is not subalgebra of  $\mathfrak{so}(9)$ , it follows that the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space constructed in this way is of type F II-II-II. This argument also shows that there is no  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type F I-II-II.

Now let  $G = E_8$  and let  $\sigma$  be an involution of type E IX as given in Example 3.1 and let  $\mathfrak{g}^\sigma$  be the Lie algebra of the fixed point set of  $i_g$ . Consider the subalgebra  $\mathfrak{h} \subset \mathfrak{g} = \mathfrak{e}_8$  isomorphic to  $\mathfrak{su}(3) + \mathfrak{e}_6$  given by the diagram below.



Let  $W$  be the Weyl group of  $\mathfrak{e}_8$  with respect to  $\mathfrak{g}_0 = \mathfrak{t}$ . The Weyl groups of  $\mathfrak{g}^\sigma$  and of  $\mathfrak{h}$  are subgroups of  $W$  in a natural way. Comparing the two diagrams corresponding to  $\mathfrak{g}^\sigma$  and  $\mathfrak{h}$ , respectively, we see that there is at least one root space of  $\mathfrak{g}$  which is contained in the  $\mathfrak{su}(3)$ -summand of  $\mathfrak{h}$ , but not in  $\mathfrak{g}^\sigma$ . Hence there is a Weyl group element  $w \in W \subset \text{Aut}(\mathfrak{g})$  whose action on  $\mathfrak{g}$  leaves the  $\mathfrak{e}_6$ -summand of  $\mathfrak{h}$  fixed, but which does not leave  $\mathfrak{g}^\sigma$  invariant. This implies that there is an involution

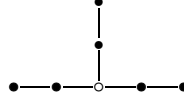
$\tau \neq \sigma$  such that  $\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$  contains a subalgebra isomorphic to  $\mathfrak{e}_6$ . By a dimension count, this shows the existence of a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E IX-IX-IX with  $\mathfrak{k} \cong \mathfrak{e}_6 + \mathbb{R} + \mathbb{R}$ .

Now consider  $\mathfrak{g} = \mathfrak{e}_7$  and let  $\sigma$  be an involution as given by the diagram E VI in Example 3.1. Consider the subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  isomorphic to  $\mathfrak{su}(6) + \mathfrak{su}(3)$  given by the following diagram.



With an analogous argument as above we may construct a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space such that  $\mathfrak{k}$  contains a subalgebra isomorphic to  $\mathfrak{su}(6)$ . Using Table 4 we see that the corresponding  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type E VI-VI-VI with  $\mathfrak{k} \cong \mathfrak{u}(6) + \mathbb{R}$ .

For  $G = E_6$ , consider the subgroup locally isomorphic to  $SU(3) \cdot SU(3) \cdot SU(3)$  given by the following diagram.

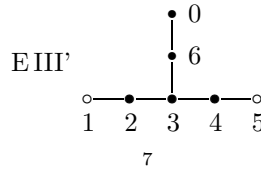


Let  $i_g$  be an inner involution corresponding to the diagram E II from Example 3.1. Then there is a Weyl group element  $w = k Z_G(T)$  such that the involution  $i_h$  with  $h = k g k^{-1}$  is such that the Lie algebra of the common fixed point set of  $i_g$  and  $i_h$  contains a subalgebra isomorphic to  $\mathfrak{su}(3) + \mathfrak{su}(3) + \mathbb{R} + \mathbb{R}$ . It follows from Table 4 that we have constructed a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E II-II-II with  $\mathfrak{k} = \mathfrak{su}(3) + \mathfrak{su}(3) + \mathbb{R} + \mathbb{R}$ .

**Example 3.3.** Here we combine two involutions of Type 3 and Type 1, respectively. Thus let  $\mathfrak{g}$  be a complex semisimple Lie algebra and let  $\mathfrak{g}_0 \subset \mathfrak{g}$  be a Cartan subalgebra. Let  $\sigma$  be an involution of Type 3 on  $\mathfrak{g}$ . Let  $\tau$  be an inner involution given as conjugation by an element  $t = \exp X$ , where  $X \in \mathfrak{g}_0$ . Then the connected component of  $Z_G(\{t\})$  is the connected Lie subgroup of  $G$  corresponding to  $\mathfrak{g}_0 + \sum_{\alpha \in S} \mathfrak{g}_\alpha$ , where  $S$  is a certain subset of the root system  $\Phi$  of  $\mathfrak{g}$  with the property that for each  $\alpha \in S$  we have also  $-\alpha \in S$ . Obviously, the two involutions commute and the involution induced by  $\sigma$  on the fixed point set  $\mathfrak{g}^\tau$  of  $\tau$  is the automorphism which acts as minus identity on  $\mathfrak{g}_0$  and sends each root  $\alpha \in S$  of  $\mathfrak{g}^\tau$  to its negative. Moreover, the fixed point set of  $\sigma\tau$  will be isomorphic to the fixed point set of  $\sigma$ , since the corresponding symmetric spaces will have isomorphic Satake diagrams. Hence the data of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space obtained in this way can be immediately deduced from the classification of ordinary symmetric spaces. This shows the existence of the exceptional  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of types E I-I-II, E I-I-III, E V-V-V, E V-V-VI, E V-V-VII, E VIII-VIII-VIII, E VIII-VIII-IX, F I-I-I, F I-I-II, G as given in Table 1.

**Example 3.4.** Let now  $G = E_6$  and choose a Cartan subalgebra  $\mathfrak{g}_0 \subset \mathfrak{e}_6$ . Let  $\sigma$  be an involution of Type 2, i.e. E IV and let  $\tau$  be an involution of Type 3, i.e. E I. Then the two involutions obviously commute. A dimension count using Tables 3 and 4 shows that  $\sigma\tau$  is of type E II and hence the corresponding  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type E I-II-IV with  $\mathfrak{k} \cong \mathfrak{sp}(3) + \mathfrak{sp}(1)$ .

**Example 3.5.** Let  $G = E_6$ . Let  $\sigma$  be the involution of Type 2 given by the permutation of the simple roots  $\alpha_1 \mapsto \alpha_5$ ,  $\alpha_2 \mapsto \alpha_4$ ,  $\alpha_3 \mapsto \alpha_3$ ,  $\alpha_4 \mapsto \alpha_2$ ,  $\alpha_5 \mapsto \alpha_1$ ,  $\alpha_6 \mapsto \alpha_6$  and let  $\tau$  be the involution of Type 1 given by the following diagram, where the numbering of the roots  $\alpha_0, \dots, \alpha_6$  is given.



Type	$d$	E III-III	-14	E VII-VII	-25
E I-I	6	E III-IV	-20	E VIII-VIII	8
E I-II	4	E IV-IV	-26	E VIII-IX	-8
E I-III	-4	E V-V	7	E IX-IX	-24
E I-IV	-10	E V-VI	1	F I-I	4
E II-II	2	E V-VII	-9	F I-II	-8
E II-III	-6	E VI-VI	-5	F II-II	-20
E II-IV	-12	E VI-VII	-15	G	2

TABLE 3. Pairs of involutions on exceptional groups.

Then obviously  $\sigma$  and  $\tau$  commute and since  $\sigma\tau$  is an outer automorphism of  $E_6$ , it follows from Lemma 4.1 that we have constructed a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E III-IV-IV with  $\mathfrak{k} \cong \mathfrak{so}(9)$ .

**Example 3.6.** Let  $G = E_7$ . By [15], Thm. 3.1, there is an isotropy irreducible homogeneous space  $E_7/L$ , where  $L$  is locally isomorphic to  $\mathrm{Sp}(1) \cdot F_4$  such that the 78-dimensional isotropy representation  $\rho$  is equivalent to the tensor product of the adjoint representation of  $\mathrm{Sp}(1)$  and the 26-dimensional irreducible representation of  $F_4$ . In particular, we may view the isotropy representation as a representation of  $\mathrm{SO}(3) \times F_4$  and choose two elements  $g, h \in L$  such that the action of  $\rho(g)$  and  $\rho(h)$  is given by  $\mathrm{diag}(-1, -1, +1) \otimes e$  and  $\mathrm{diag}(1, -1, -1) \otimes e$ , respectively, where  $e$  stands for the identity element of  $F_4$ . Then the involutions  $i_g$  and  $i_h$  generate a subgroup  $\Gamma \subset \mathrm{Aut}(E_7)$  isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  whose fixed point set has Lie algebra  $\mathfrak{k} \cong \mathfrak{f}_4$ . Since the fixed point sets of  $i_g$ ,  $i_h$  and  $i_{gh}$  are all 79-dimensional, it follows that we have constructed a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E VII-VII-VII.

#### 4. THE CLASSIFICATION IN THE EXCEPTIONAL CASE

**Lemma 4.1.** *There are no  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces of type F I-II-II, E I-III-IV, E II-IV-IV, E V-VI-VI, E VI-VI-VII E V-VII-VII.*

*Proof.* The non-existence of type F I-II-II was already shown in Example 3.2. The non-existence of type E I-III-IV or E II-IV-IV spaces follows immediately from Tables 3 and 4. For a pair of involutions of type E VI-VI on  $E_7$  we have  $d = -5$ , see Table 3. It follows from Table 4 that there is no symmetric subalgebra  $\mathfrak{k}$  of  $\mathfrak{h} = \mathfrak{su}(8)$  or  $\mathfrak{h} = \mathfrak{e}_6 + \mathbb{R}$  such that  $c(\mathfrak{su}(8), \mathfrak{k}) = -5$ ; this shows there are no  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces of type E V-VI-VI or E VI-VI-VII. Now consider a pair of involutions of type E VII-VII on  $E_7$ ; by Table 3 we have  $d = -25$  and it follows from Table 4 that  $\mathfrak{k} \cong \mathfrak{so}(10) + \mathbb{R} + \mathbb{R}$  or  $\mathfrak{k} \cong \mathfrak{f}_4$ ; since neither occurs as a subalgebra of  $\mathfrak{su}(8)$ , we have proved the non-existence of a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E V-VII-VII.  $\square$

**Remark 4.2.** In each row of Table 1, only the isomorphism type of the subalgebra  $\mathfrak{k} \subset \mathfrak{g}$  and the type of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space (i.e. the entry in the first column defines the isomorphism type of the three symmetric subalgebras  $\mathfrak{g}^\sigma, \mathfrak{g}^\tau, \mathfrak{g}^{\sigma\tau}$  of  $\mathfrak{g}$ ) is given. However, this information is sufficient to determine the conjugacy class of the subalgebra  $\mathfrak{k} \subset \mathfrak{g}$  in most cases, since  $\mathfrak{k} \subset \mathfrak{h}$  is also symmetric subalgebra for  $\mathfrak{h} = \mathfrak{g}^\sigma, \mathfrak{g}^\tau, \mathfrak{g}^{\sigma\tau}$ : Assume that  $\mathfrak{g}$  is a compact Lie algebra and let  $\mathfrak{k} \subset \mathfrak{h} \subset \mathfrak{g}$  and  $\mathfrak{k}' \subset \mathfrak{h}' \subset \mathfrak{g}$  be such that all inclusions are of symmetric subalgebras and such that  $\mathfrak{k} \cong \mathfrak{k}'$ ,  $\mathfrak{h} \cong \mathfrak{h}'$ . Then there is an automorphism  $\eta \in \mathrm{Aut}(\mathfrak{g})$  such that  $\eta(\mathfrak{h}') = \mathfrak{h}$ . In most cases (cf. [12], 3.1) there is an inner automorphism  $\theta_0$  of  $\mathfrak{h}$  such that  $\theta_0(\eta(\mathfrak{k}')) = \mathfrak{k}$ . Then obviously  $\theta_0$  can be extended to an inner automorphism  $\theta$  of  $\mathfrak{g}$  such that  $\theta \circ \eta(\mathfrak{k}') = \mathfrak{k}$ .



$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{k}$	$c(\mathfrak{h}, \mathfrak{k})$
$\mathfrak{e}_6$	$\mathfrak{sp}(4)$	$\mathfrak{sp}(4-n) + \mathfrak{sp}(n)$ $n = 1, 2$	$-12, -4$
$\mathfrak{e}_6$	$\mathfrak{sp}(4)$	$\mathfrak{su}(4) + \mathbb{R}$	$4$
$\mathfrak{e}_6$	$\mathfrak{su}(6) + \mathfrak{sp}(1)$	$\mathfrak{s}(\mathfrak{u}(6-n) + \mathfrak{u}(n)) + \mathbb{R}$ $n = 0, \dots, 3$	$-34, -14,$ $-2, 2$
$\mathfrak{e}_6$	$\mathfrak{su}(6) + \mathfrak{sp}(1)$	$\mathfrak{s}(\mathfrak{u}(6-n) + \mathfrak{u}(n)) + \mathfrak{sp}(1)$ $n = 1, \dots, 3$	$-18, -6, -2$
$\mathfrak{e}_6$	$\mathfrak{su}(6) + \mathfrak{sp}(1)$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$-10$
$\mathfrak{e}_6$	$\mathfrak{su}(6) + \mathfrak{sp}(1)$	$\mathfrak{sp}(3) + \mathbb{R}$	$-6$
$\mathfrak{e}_6$	$\mathfrak{su}(6) + \mathfrak{sp}(1)$	$\mathfrak{so}(6) + \mathfrak{sp}(1)$	$2$
$\mathfrak{e}_6$	$\mathfrak{su}(6) + \mathfrak{sp}(1)$	$\mathfrak{so}(6) + \mathbb{R}$	$6$
$\mathfrak{e}_6$	$\mathfrak{so}(10) + \mathbb{R}$	$\mathfrak{so}(10-n) + \mathfrak{so}(n)$ $n = 0, \dots, 5$	$-44, -26, -12,$ $-2, 4, 6$
$\mathfrak{e}_6$	$\mathfrak{so}(10) + \mathbb{R}$	$\mathfrak{so}(10-n) + \mathfrak{so}(n) + \mathbb{R}$ $n = 1, \dots, 5$	$-28, -14,$ $-4, 2, 4$
$\mathfrak{e}_6$	$\mathfrak{so}(10) + \mathbb{R}$	$\mathfrak{u}(5) + \mathbb{R}$	$-6$
$\mathfrak{e}_6$	$\mathfrak{so}(10) + \mathbb{R}$	$\mathfrak{u}(5)$	$-4$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$4$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathfrak{so}(9)$	$-20$
$\mathfrak{e}_7$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(8-n) + \mathfrak{u}(n))$ $n = 1, \dots, 4$	$-35, -15,$ $-3, 1$
$\mathfrak{e}_7$	$\mathfrak{su}(8)$	$\mathfrak{sp}(4)$	$-9$
$\mathfrak{e}_7$	$\mathfrak{su}(8)$	$\mathfrak{so}(8)$	$7$
$\mathfrak{e}_7$	$\mathfrak{so}(12) + \mathfrak{sp}(1)$	$\mathfrak{so}(12-n) + \mathfrak{so}(n) + \mathbb{R}$ $n = 0, \dots, 6$	$-65, -43, -25,$ $-11, -1, 5, 7$
$\mathfrak{e}_7$	$\mathfrak{so}(12) + \mathfrak{sp}(1)$	$\mathfrak{so}(12-n) + \mathfrak{so}(n) + \mathfrak{sp}(1),$ $n = 1, \dots, 6$	$-47, -29,$ $-15, -5, 1, 3$

TABLE 4. Symmetric subalgebras of symmetric subalgebras of simple exceptional compact Lie algebras.

This shows that for most cases the subalgebra  $\mathfrak{k}$  is uniquely defined by the data in Table 1. The only exception occurs for the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E VI-VI-VI with  $\mathfrak{k} \cong \mathfrak{u}(6) + \mathbb{R}$ . Indeed, there are two conjugacy classes of subalgebras isomorphic to  $\mathfrak{a}_5$  in  $\mathfrak{e}_7$ . These two classes are denoted as  $[A_5]'$  and  $[A_5]''$  in [5]. They are given by the two diagrams below.



Note that these two subalgebras of  $\mathfrak{e}_7$  are both contained in  $\mathfrak{so}(12) \subset \mathfrak{e}_7$  and, as subalgebras of  $\mathfrak{so}(12)$ , they are conjugate by an outer automorphism of  $\mathfrak{so}(12)$ ; however, this automorphism cannot be extended to an automorphism of  $\mathfrak{e}_7$ . It follows from Tables 25 and 26, p. 204, 207 in [5] that only the subalgebra  $[A_5]''$  gives rise to a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space.

$\mathfrak{e}_7$	$\mathfrak{so}(12) + \mathfrak{sp}(1)$	$\mathfrak{u}(6) + \mathfrak{sp}(1)$	$-9$
$\mathfrak{e}_7$	$\mathfrak{so}(12) + \mathfrak{sp}(1)$	$\mathfrak{u}(6) + \mathbb{R}$	$-5$
$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{R}$	$\mathfrak{sp}(4) + \mathbb{R}$	$5$
$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{R}$	$\mathfrak{sp}(4)$	$7$
$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{R}$	$\mathfrak{so}(10) + \mathbb{R} + \mathbb{R}$	$-15$
$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{R}$	$\mathfrak{so}(10) + \mathbb{R}$	$-13$
$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{R}$	$\mathfrak{su}(6) + \mathfrak{sp}(1) + \mathbb{R}$	$1$
$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{R}$	$\mathfrak{su}(6) + \mathfrak{sp}(1)$	$3$
$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{R}$	$\mathfrak{f}_4 + \mathbb{R}$	$-27$
$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{R}$	$\mathfrak{f}_4$	$-25$
$\mathfrak{e}_8$	$\mathfrak{so}(16)$	$\mathfrak{so}(16-n) + \mathfrak{so}(n)$ $n = 1, \dots, 8$	$-90, -64, -42,$ $-24, -10, 0, 6, 8$
$\mathfrak{e}_8$	$\mathfrak{so}(16)$	$\mathfrak{u}(8)$	$-8$
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{e}_7 + \mathbb{R}$	$-132$
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{su}(8) + \mathfrak{sp}(1)$	$4$
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{su}(8) + \mathbb{R}$	$8$
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{so}(12) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$	$-8$
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{so}(12) + \mathfrak{sp}(1) + \mathbb{R}$	$-4$
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{e}_6 + \mathbb{R} + \mathfrak{sp}(1)$	$-28$
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{e}_6 + \mathbb{R} + \mathbb{R}$	$-24$
$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$\mathfrak{sp}(3) + \mathbb{R}$	$-20$
$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$\mathfrak{sp}(2) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$	$-8$
$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$\mathfrak{sp}(2) + \mathfrak{sp}(1) + \mathbb{R}$	$-4$
$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$\mathfrak{u}(3) + \mathfrak{sp}(1)$	$0$
$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$\mathfrak{u}(3) + \mathbb{R}$	$4$
$\mathfrak{f}_4$	$\mathfrak{so}(9)$	$\mathfrak{so}(9-n) + \mathfrak{so}(n)$ $n = 1, \dots, 4$	$-20, -8, 0, 4$
$\mathfrak{g}_2$	$\mathfrak{so}(4)$	$\mathfrak{u}(2)$	$-2$
$\mathfrak{g}_2$	$\mathfrak{so}(4)$	$\mathfrak{so}(3)$	$0$
$\mathfrak{g}_2$	$\mathfrak{so}(4)$	$\mathbb{R} + \mathbb{R}$	$2$

TABLE 4. (cont.)

*Proof of Theorem 1.2.* For each entry in Table 1 we have given a construction in Section 3 thus showing the existence of a corresponding  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space. Lemma 4.1 shows that the types of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space not listed in Table 1 indeed do not exist. (Note that on  $E_6$ , the involutions of type E I and E IV are outer, while the involutions of type E II and E III are inner; in particular, there cannot be  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces of type E I-I-I or of type E I-II-II etc.)

However, it remains to be shown that the type of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space as given in Table 1 indeed determines the conjugacy class of the fixed point set  $\mathfrak{k} = \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$  except for spaces of type E VI-VI-VI. In most cases, it can be seen from Tables 3 and 4 that the conjugacy class of the subalgebra  $\mathfrak{k} \subset \mathfrak{g}$  is uniquely determined by the type of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space, i.e. in most cases, the conjugacy class of the subalgebra  $\mathfrak{k}$  is uniquely determined by the triple  $(\mathfrak{g}, \mathfrak{h}, c(\mathfrak{h}, \mathfrak{k}))$  where  $c(\mathfrak{h}, \mathfrak{k}) = d$ . There are three exceptions to this general rule:

For  $\mathfrak{g} = \mathfrak{e}_6$  we have two entries in Table 4 with  $\mathfrak{h} = \mathfrak{su}(6) + \mathfrak{sp}(1)$  and  $c(\mathfrak{h}, \mathfrak{k}) = 2$ , namely  $\mathfrak{k} = \mathfrak{su}(3) + \mathfrak{su}(3) + \mathbb{R} + \mathbb{R}$  and  $\mathfrak{k} = \mathfrak{so}(6) + \mathfrak{sp}(1)$ . For the first possibility we have already shown the existence of a corresponding  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E II-II-II. The second possibility

would imply the existence of a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E II-II-II with  $\mathfrak{k} = \mathfrak{so}(6) + \mathfrak{sp}(1) \subset \mathfrak{su}(6) + \mathfrak{sp}(1)$  such that the isotropy representation of  $\mathfrak{k}$  on  $\mathfrak{g}/\mathfrak{k}$  restricted to  $\mathfrak{so}(6)$  consists of a sum of three isotypical summands; however this is not the case according to [5], Tables 25, 26, pp. 200, 206.

There are also two entries in Table 4 with  $\mathfrak{h} = \mathfrak{su}(6) + \mathfrak{sp}(1)$  and  $c(\mathfrak{h}, \mathfrak{k}) = -6$ , namely  $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) + \mathfrak{sp}(1)$  and  $\mathfrak{k} = \mathfrak{sp}(3) + \mathbb{R}$ . For the first we have already shown the existence of a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E II-II-III in Example 3.1; the second one can be ruled out since  $\mathfrak{sp}(3) + \mathbb{R}$  is not a symmetric subalgebra of  $\mathfrak{so}(10) + \mathbb{R}$ .

Finally, for  $\mathfrak{g} = \mathfrak{e}_7$  there are two items in Table 4 with  $\mathfrak{h} = \mathfrak{so}(12) + \mathfrak{sp}(1)$  and  $c(\mathfrak{h}, \mathfrak{k}) = -5$ , namely  $\mathfrak{k} = \mathfrak{so}(8) + \mathfrak{so}(4) + \mathfrak{sp}(1)$  and  $\mathfrak{k} = \mathfrak{u}(6) + \mathbb{R}$ . We have shown the existence of a corresponding  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type E VI-VI-VI for both cases in Examples 3.1 and 3.2, respectively.  $\square$

## 5. THE CLASSIFICATION IN THE Spin(8) CASE

In [1], the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of the classical groups have been determined except for  $G \cong \text{Spin}(8)$ . This case requires special attention since the group of outer automorphisms  $\text{Out}(\text{Spin}(8)) = \text{Aut}(\text{Spin}(8))/\text{Inn}(\text{Spin}(8))$  is isomorphic to the symmetric group on three letters, whereas for all other simple compact Lie groups the group  $\text{Out}(G)$  is either trivial or isomorphic to  $\mathbb{Z}_2$ . In this section, we will classify the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces with  $G = \text{Spin}(8)$ . It was remarked in [1] that the result can be obtained from [4]. We give an independent proof, using an analogous method as in the exceptional case.

We recall [12] that all connected symmetric subgroups of  $\text{Spin}(8)$  are conjugate by some automorphism of  $\text{Spin}(8)$  to one of the groups

$$\text{Spin}(8 - n) \cdot \text{Spin}(n), \quad n = 1, 2, 3, 4.$$

However, if one considers conjugacy classes with respect to inner automorphisms, there are three conjugacy classes of symmetric subgroups locally isomorphic to one of  $\text{Spin}(7)$ ,  $\text{Spin}(6) \cdot \text{Spin}(2)$ ,  $\text{Spin}(5) \cdot \text{Spin}(3)$ , respectively, and only one conjugacy class of subgroups locally isomorphic to  $\text{Spin}(4) \cdot \text{Spin}(4)$ , cf. [12], Proposition 3.3. Hence there are ten conjugacy classes (w.r.t. inner automorphisms) of connected symmetric subgroups in  $\text{Spin}(8)$ ; under the covering map  $\text{Spin}(8) \rightarrow \text{SO}(8)$ , they correspond to the conjugacy classes (w.r.t. inner automorphisms) of connected (locally) symmetric subgroups in  $\text{SO}(8)$  given by  $\text{SO}(7)$ ,  $\text{Spin}(7)^+$ ,  $\text{Spin}(7)^-$ ,  $\text{SO}(6) \times \text{SO}(2)$ ,  $\text{U}(4)$ ,  $\alpha(\text{U}(4))$ ,  $\text{SO}(5) \times \text{SO}(3)$ ,  $\text{Sp}(2) \cdot \text{Sp}(1)$ ,  $\alpha(\text{Sp}(2) \cdot \text{Sp}(1))$ ,  $\text{SO}(4) \times \text{SO}(4)$ , where  $\alpha = i_{\text{diag}(-1, 1, \dots, 1)}$ .

$G^\sigma$	$G^\tau$	$d$
$\text{Spin}(7)$	$\text{Spin}(7)$	$-14$
$\text{Spin}(7)$	$\text{Spin}(6) \cdot \text{Spin}(2)$	$-9$
$\text{Spin}(7)$	$\text{Spin}(5) \cdot \text{Spin}(3)$	$-6$
$\text{Spin}(7)$	$\text{Spin}(4) \cdot \text{Spin}(4)$	$-5$
$\text{Spin}(6) \cdot \text{Spin}(2)$	$\text{Spin}(6) \cdot \text{Spin}(2)$	$-4$
$\text{Spin}(6) \cdot \text{Spin}(2)$	$\text{Spin}(5) \cdot \text{Spin}(3)$	$-1$
$\text{Spin}(6) \cdot \text{Spin}(2)$	$\text{Spin}(4) \cdot \text{Spin}(4)$	$0$
$\text{Spin}(5) \cdot \text{Spin}(3)$	$\text{Spin}(5) \cdot \text{Spin}(3)$	$2$
$\text{Spin}(5) \cdot \text{Spin}(3)$	$\text{Spin}(4) \cdot \text{Spin}(4)$	$3$
$\text{Spin}(4) \cdot \text{Spin}(4)$	$\text{Spin}(4) \cdot \text{Spin}(4)$	$4$

TABLE 5. Pairs of involutions on  $\text{Spin}(8)$ .

**Example 5.1.** We may construct standard examples of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces for  $\mathfrak{g} = \mathfrak{so}(8)$  using a pair of commuting involutions both given by conjugation with diagonal matrices of the form  $\text{diag}(\pm 1, \dots, \pm 1)$ . In this way obtain  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces with  $\mathfrak{k} = \mathfrak{so}(n_1) + \mathfrak{so}(n_2) + \mathfrak{so}(n_3)$ ,  $n_1 + n_2 + n_3 = 8$ ,  $n_i \geq 1$  or  $\mathfrak{k} = \mathfrak{so}(n_1) + \mathfrak{so}(n_2) + \mathfrak{so}(n_3) + \mathfrak{so}(n_4)$ ,  $n_1 + n_2 + n_3 + n_4 = 8$ ,  $n_i \geq 1$ .

**Example 5.2.** Now assume the subalgebra  $\mathfrak{so}(6) + \mathfrak{so}(2) \cong \mathfrak{u}(4) \subset \mathfrak{so}(8)$  is embedded as

$$\left\{ \begin{pmatrix} A & -B^t \\ B & A \end{pmatrix} \mid A + iB \in \mathfrak{u}(4) \right\}.$$

We may define a pair of commuting involutions of  $\mathfrak{so}(8)$  using conjugation with the matrices

$$J = \begin{pmatrix} & -I \\ I & \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix},$$

respectively, where  $I$  is the  $4 \times 4$ -identity matrix and  $D$  is either  $\text{diag}(-1, 1, 1, 1)$  or  $\text{diag}(-1, -1, 1, 1)$ . We obtain  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces with  $\mathfrak{k} = \mathfrak{u}(1) + \mathfrak{u}(3)$  or  $\mathfrak{k} = \mathfrak{u}(2) + \mathfrak{u}(2)$ . Note however that the subalgebra  $\mathfrak{u}(2) + \mathfrak{u}(2)$  of  $\mathfrak{so}(8)$  is conjugate to  $\mathfrak{so}(4) + \mathfrak{so}(2) + \mathfrak{so}(2)$  via an outer automorphism.

*Proof of Theorem 1.3.* Assume  $\sigma$  and  $\tau$  are two commuting involutions of  $\text{Spin}(8)$ . Then the  $\mathfrak{g}^{\sigma\tau}$  is a symmetric subalgebra of  $\mathfrak{g} = \mathfrak{so}(8)$  by Proposition 2.2. Hence we may assume (by applying an automorphism of  $\mathfrak{so}(8)$ ) that  $\mathfrak{g}^{\sigma\tau}$  is one of the subalgebras  $\mathfrak{h}$  of  $\mathfrak{so}(8)$  as given Table 6. Since  $\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau \subset \mathfrak{h}$  is a symmetric subalgebra of  $\mathfrak{h}$ , it is conjugate (by some automorphism  $\theta_0$  of  $\mathfrak{h}$ ) to one of the subalgebras  $\mathfrak{k}$  as from Table 6 such that  $c(\mathfrak{h}, \mathfrak{k}) = d(\sigma, \tau)$ .

Now we show that any automorphism  $\theta_0$  of  $\mathfrak{h}$  can be extended to an automorphism  $\theta$  of  $\mathfrak{so}(8)$  such that  $\theta(\mathfrak{h}) = \mathfrak{h}$  and  $\theta|_{\mathfrak{h}} = \theta_0$ : This is obvious for inner automorphisms of  $\mathfrak{h}$  and there is only one case where  $\mathfrak{h}$  has outer automorphisms, namely  $\mathfrak{h} = \mathfrak{so}(4) + \mathfrak{so}(4) \cong 4 \cdot \mathfrak{a}_1$ . After choosing a Cartan subalgebra of  $\mathfrak{g}$  contained in  $\mathfrak{h}$ , and a set of simple roots  $\alpha_1, \dots, \alpha_4$  we can define diagram automorphisms which act by permuting the simple roots  $\alpha_1, \alpha_2, \alpha_3$  and hence by permuting the corresponding three of the four  $\mathfrak{a}_1$ -summands of  $\mathfrak{h}$ . In addition, consider the inner automorphism  $i_g$  with  $g = \begin{pmatrix} & I \\ I & \end{pmatrix}$ ,  $I = \text{diag}(1, 1, 1, 1)$ . This acts on  $\mathfrak{h} = \mathfrak{so}(4) + \mathfrak{so}(4)$  by interchanging the two  $\mathfrak{so}(4)$ -summands. These automorphisms together with the inner automorphisms of  $\mathfrak{h}$  extended to  $\mathfrak{g}$  generate a subgroup of  $\text{Aut}(\mathfrak{g})$  isomorphic to  $\text{Aut}(\mathfrak{h})$ .

Hence there is an automorphism  $\varphi$  of  $\mathfrak{so}(8)$  such that  $\varphi(\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau)$  is one the subalgebras  $\mathfrak{k} \subset \mathfrak{so}(8)$  as given in Table 6. Conversely, it is straightforward to see that for each candidate  $(\mathfrak{h}, \mathfrak{k})$  one can construct a corresponding  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space using Examples 5.1 or 5.2.  $\square$

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$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{k}$	$c(\mathfrak{h}, \mathfrak{k})$
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(7-n) + \mathfrak{so}(n)$ $n = 1, 2, 3$	$-9, -1, 3$
$\mathfrak{so}(8)$	$\mathfrak{so}(6) + \mathfrak{so}(2)$	$\mathfrak{so}(6-n) + \mathfrak{so}(n) + \mathfrak{so}(2)$ $n = 1, 2, 3$	$-6, 0, 2$
$\mathfrak{so}(8)$	$\mathfrak{so}(6) + \mathfrak{so}(2)$	$\mathfrak{so}(6-n) + \mathfrak{so}(n)$ $n = 0, 1, 2, 3$	$-14, -4, 2, 4$
$\mathfrak{so}(8)$	$\mathfrak{so}(6) + \mathfrak{so}(2)$	$\mathfrak{u}(3) + \mathfrak{u}(1)$	$-4$
$\mathfrak{so}(8)$	$\mathfrak{so}(6) + \mathfrak{so}(2)$	$\mathfrak{u}(3)$	$-2$
$\mathfrak{so}(8)$	$\mathfrak{so}(5) + \mathfrak{so}(3)$	$\mathfrak{so}(5-n) + \mathfrak{so}(n) + \mathfrak{so}(3)$ $n = 1, 2$	$-5, -1$
$\mathfrak{so}(8)$	$\mathfrak{so}(5) + \mathfrak{so}(3)$	$\mathfrak{so}(5-n) + \mathfrak{so}(n) + \mathfrak{so}(2)$ $n = 0, 1, 2$	$-9, -1, 3$
$\mathfrak{so}(8)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$	$(4-n) \cdot \mathfrak{sp}(1) + n \cdot \mathfrak{so}(2)$ $n = 1, 2, 3, 4$	$-8, -4, 0, 4$
$\mathfrak{so}(8)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$	$\mathfrak{so}(3) + \mathfrak{so}(4)$	$-6$
$\mathfrak{so}(8)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$	$\mathfrak{so}(3) + \mathfrak{so}(3)$	$0$
$\mathfrak{so}(8)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$	$\mathfrak{so}(3) + \mathfrak{u}(2)$	$-2$
$\mathfrak{so}(8)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$	$\mathfrak{so}(3) + \mathfrak{so}(2) + \mathfrak{so}(2)$	$2$

TABLE 6. Symmetric subalgebras of symmetric subalgebras of  $\mathfrak{so}(8)$ .

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